

CONSTRAINED WILLMORE AND CMC TORI IN THE 3-SPHERE

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ABSTRACT. Constrained Willmore surfaces are critical points of the Willmore functional under conformal variations. As shown in [5] one can associate to any conformally immersed constrained Willmore torus f a compact Riemann surface Σ , such that f can be reconstructed in terms of algebraic data on Σ . Particularly interesting examples of constrained Willmore tori are the tori with constant mean curvature (CMC) in a 3-dimensional space form. It is shown in [14] and in [16] that the spectral curves of these tori are hyperelliptic. In this paper we show under mild conditions that a constrained Willmore torus f in S^3 is a CMC torus in a 3-dimensional space form if its spectral curve has the structure of a CMC spectral curve.

1. INTRODUCTION

The study of constant mean curvature (CMC) tori is classical in differential geometry. Hopf conjectured that the only compact CMC surface in \mathbb{R}^3 would be the round sphere. This was proven by Alexandrov [1] in the case of embeddings in 1956. The first counterexample was found by Wente [19] in 1986 in the class of immersed tori. In fact there exist a great variety of immersed CMC tori in 3-dimensional space forms. Hitchin [14] and Pinkall and Sterling [16] found independently that CMC tori in 3-space form an integrable system, i.e., one can associate to every CMC torus a Riemann surface Σ - the spectral curve - and the immersion can be reconstructed in terms of algebraic data on Σ . For CMC tori Σ is hyperelliptic. Based on these results Bobenko [3] constructed all CMC tori in 3-dimensional space forms explicitly in terms of theta functions.

On the other hand, constrained Willmore surfaces are critical points of the Willmore functional $\int (H^2 + 1) dA$ under conformal variations, see [8] and [15]. Compact constrained Willmore minimizers can be viewed as optimal realizations of the underlying Riemann surface in 3-space. The Willmore functional is invariant under conformal transformations of the ambient space, thus we do not distinguish between Möbius equivalent surfaces. Examples of constrained Willmore tori are CMC tori in a space form. Another related class of constrained Willmore tori were found by Babich and Bobenko [2]. These tori consists of two non compact components, each of them has constant mean curvature in H^3 . The components are glued together over the infinity boundary of the hyperbolic space. Its is shown in [5] that constrained Willmore tori

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in S^3 also form an integrable system. The spectral curve Σ is here either a 4-fold covering or a double covering of \mathbb{CP}^1 .

Two natural questions are whether both definitions of a spectral curve coincide in the case of CMC tori and whether the structure of the spectral curve already determines the immersion type. The first question is answered in the affirmative by [5]. The constrained Willmore spectral curve of a CMC torus corresponds to the harmonic map spectral curve of its Gauß map. The second assertion is not true in general. Counter examples are given by a simple factor dressing of certain homogenous tori, viewed in the constrained Willmore setup. Simple factor dressing preserves the spectral curve but the resulting torus is in general not CMC. We rule out this case by demanding the uniqueness of the reconstruction of the surface up to Möbius transformations from the spectral data.

In this paper we first recall the definition of the spectral curve Σ for CMC tori due to [4] and [14]. This spectral curve is hyperelliptic and its genus g is the geometric genus of the immersion. For the reconstruction it is more convenient to consider a (possibly singular) curve $\tilde{\Sigma}$ for which Σ is the normalization. The genus p of $\tilde{\Sigma}$ is the arithmetic genus of the immersion, see [14]. The reconstruction of the immersion is then given by the Sym-Bobenko formula [4].

Then we briefly discuss the definition of the spectral curve for general constrained Willmore tori introduced in [5]. The so defined spectral curve is shown in [5] to coincide with the one defined in [6] for conformally immersed tori and is by definition a smooth curve. Thus we can use Theorem 4.2 of [6] for the reconstruction of a constrained Willmore immersion. We call a immersion simple if this reconstruction is unique up to Möbius transformations.

For CMC tori simple means that the arithmetic spectral genus equals the geometric spectral genus of the immersion. The main theorem of the paper is the following.

Theorem 1. *A simple constrained Willmore torus in S^3 is a CMC torus in a space form if and only if its spectral curve has the structure of a CMC spectral curve.*

A necessary condition is that a certain anti-holomorphic involution on Σ has fixpoints. This excludes the tori constructed by Babich and Bobenko [2] mentioned before and is always satisfied if the spectral genus is even. In the case of spectral genus $g \leq 2$ the following holds: It is well known that simple tori of spectral genus 0 are always homogenous. We show that a simple immersion $f : T^2 \rightarrow S^3$ of spectral genus 1 is always equivariant, i.e., it has a 1-parameter group of Möbius symmetries. Thus it is always CMC in a space form by [13]. If the spectral genus is 2 we show that simple constrained Willmore tori are either equivariant or CMC in a space form.

2. SPECTRAL CURVE FOR CMC TORI

This definition of the spectral curve is due to [14] and [4]. We define it only for the case of CMC tori in S^3 . Let $f : T^2 \rightarrow S^3$ be a conformally immersed torus with mean curvature H and let $\alpha = f^{-1}df$ be the corresponding Maurer-Cartan form.

We denote by W the pull back of the spinor bundle of S^3 . This is a complex rank 2 vector bundle with a quaternionic structure j and a symplectic form $\hat{\omega}$.

Let $\alpha = \alpha' + \alpha''$ be the splitting of α into its complex linear and complex anti-linear part. Then we can associate to f a family of connections

$$(2.1) \quad \nabla^\lambda = \nabla + \frac{1}{2}(1 + \lambda^{-1})(1 + iH)\alpha' + \frac{1}{2}(1 + \lambda)(1 - iH)\alpha'', \quad \lambda \in \mathbb{C}_*,$$

where ∇ is the trivial connection on W . This family of connections is flat if and only if $H \equiv \text{const}$ in S^3 and has the symmetry

$$\nabla^{\bar{\lambda}^{-1}} = j^{-1} \nabla^\lambda j.$$

Thus for $\lambda \in S^1$ the connection ∇^λ is unitary. The following theorem shows how to reconstruct CMC immersions in a 3-dimensional space forms from this associated family of flat connections. Moreover, it shows that we can restrict ourselves without loss of generality to the associated family for minimal surfaces in S^3 given by

$$(2.2) \quad \nabla^\lambda = \nabla + \frac{1}{2}(1 + \lambda^{-1})\alpha' + \frac{1}{2}(1 + \lambda)\alpha'', \quad \lambda \in \mathbb{C}_*,$$

where ∇ is a unitary connection on W .

Theorem 2 ([4]). *Let ∇^λ be a family of flat connections on W given in (2.2) and $\lambda_0, \lambda_1 \in \mathbb{C}$. Further let X_λ , $\det X_\lambda = 1$, be a parallel frame of ∇^λ .*

For two distinct $\lambda_0, \lambda_1 \in S^1$ the map f given by

$$(2.3) \quad f = X_{\lambda_0}^{-1} X_{\lambda_1}$$

is well defined on \mathbb{C} and has constant mean curvature $H = i \frac{\lambda_0 + \lambda_1}{\lambda_0 - \lambda_1}$ in S^3

If $\lambda_0 = \lambda_1 \in S^1$ then

$$(2.4) \quad f = X_{\lambda_0}^{-1} \frac{\partial X}{\partial \lambda} \Big|_{\lambda_0}$$

is a CMC 1 surface in \mathbb{R}^3 .

And for $\lambda_1 = \bar{\lambda}_0^{-1}$ and $|\lambda_0|^2 < 1$, f given by

$$(2.5) \quad f = X_{\lambda_0}^{-1} X_{\lambda_1}$$

is CMC in H^3 with mean curvature $H = \frac{1 + |\lambda_0|^2}{1 - |\lambda_0|^2}$.

Further, all CMC tori in 3-space can be obtained this way.

Remark 1. The formulas above are called the Sym-Bobenko formulas and the λ_i are called Sym points. By choosing $\lambda_0 = -\lambda_1 \in S^1$ we obtain a minimal immersion in S^3 . Thus in order to obtain all CMC tori in 3-dimensional space forms it is sufficient to write down all possible 1-parameter families of flat connections of the form (2.2). The first step is to define a spectral curve.

Let $T^2 = \mathbb{C}/\Gamma$ and let $H_x^\lambda(\gamma)$ denote the holonomy of the connection ∇^λ along $\gamma \in \Gamma$ with respect to the base point $x \in T^2$. We choose γ to be one of the generators of Γ . By [14] the holonomy $H_x^\lambda(\gamma)$ is generically diagonalizable and has distinct eigenvalues.

The spectral curve Σ is now the normalization and compactification of the analytic variety

$$\{(\eta, \lambda) \in \mathbb{C}_* \times \mathbb{C}_* \mid f(\eta, \lambda) = 0\} \quad \text{with} \quad f(\eta, \lambda) = \det(H_x^\lambda(\gamma) - \eta Id).$$

Since $H_p^\lambda(\gamma)$ is 2×2 , Σ is hyperelliptic and we have a hyperelliptic involution σ on Σ . Further, an anti-holomorphic involution ρ is given by the quaternionic structure j . By [14] the spectral curve of a minimal immersion is branched over $\lambda = 0$ and $\lambda = \infty$. Therefore we obtain

$$\Sigma : \eta^2 = \lambda \prod_{i=1}^g \bar{q}_i^{-1} (\lambda - q_i) (\lambda - \bar{q}_i^{-1}),$$

where $q_i \neq 0, \infty$, and \bar{q}_i^{-1} are the odd order roots of the function $f(\eta, \lambda)$ without multiplicity. The change of the base point $x \in T^2$ corresponds to a conjugation of the holonomy matrix. Thus the eigenvalues and hence the spectral curve is independent of $x \in T^2$. Further, since Γ is a lattice, Σ is also independent of the choice of $\gamma \in \Gamma$.

Next we want to define a T^2 -family of complex holomorphic line bundles over the spectral curve and collect all properties these line bundles have. For a fixed $x \in T^2$ let \mathcal{L}_x denote the line bundle over Σ given by the eigenspace of the holonomy. To be more explicit, at a generic point $(\eta, \lambda) \in \Sigma$, where $H_x^\lambda(\gamma)$ has 2 distinct eigenvalues, we define the fiber $\mathcal{L}_x|_{(\eta, \lambda)}$ to be the 1-dimensional eigenspace of $H_x^\lambda(\gamma)$ with respect to the eigenvalue η . The bundle is well-defined and extends holomorphically to a line bundle over all Σ .

Proposition 1 ([14], section 7). *Let $x_0 \in T^2$. Then map*

$$\Psi : T^2 \rightarrow \text{Jac}(\Sigma), x \mapsto \mathcal{L}_x L_{x_0}^{-1}$$

is a group homomorphism.

In order to make the map Ψ explicit, it is necessary to compute the degree of \mathcal{L}_x . Let σ denote the hyperelliptic involution on Σ . Then the bundles \mathcal{L}_x and $\sigma^* \mathcal{L}_x$ are subbundles of the trivial bundle $\Sigma \times W_x$. Since W is the pull-back of the spin bundle of S^3 , there exists a symplectic form $\hat{\omega}$ on W . Therefore the evaluation of the symplectic form $\hat{\omega}_x$ on $\mathcal{L}_x \otimes \sigma^* \mathcal{L}_x$ defines a holomorphic map to \mathbb{C} . Thus $\hat{\omega}_x$ is a section in the line bundle $\mathcal{L}_x^* \otimes (\sigma^* \mathcal{L}_x)^*$ over \mathbb{CP}^1 . It vanishes exactly at those points, where \mathcal{L}_x and $\sigma^* \mathcal{L}_x$ coincides. Therefore the zeros of $\hat{\omega}_x$ do not depend on x . Obviously $\hat{\omega}_x$ vanishes at branch points of Σ , thus the degree of \mathcal{L}_x^* is at least $g + 1$.

Definition. Let $f : T^2 \rightarrow S^3$ be a conformal immersion of constant mean curvature and Σ its spectral curve. The genus g of Σ is called the geometric spectral genus of f . The arithmetic spectral genus p of f is the genus of the possibly singular curve

$$\tilde{\Sigma} : \eta^2 = \lambda \prod_{i=1}^p \bar{q}_i^{-1} (\lambda - q_i) (\lambda - \bar{q}_i^{-1}),$$

where the q_i 's are the zeros of the symplectic form $\hat{\omega}$ counted with multiplicity.

Proposition 2 ([14], Theorems 8.1 and 7.15). *The line bundle \mathcal{L}_x^* , $x \in T^2$, has degree $p + 1$, where p is the arithmetic spectral genus of f . Further, the bundle \mathcal{L}_x is non special for all $x \in T^2$, i.e., $K\mathcal{L}_x^*$ has no holomorphic sections.*

For a given spectral curve and a family of line bundles $\Psi(T^2)$ compatible with all involutions, we can reconstruct by [14] an associated family of flat connections of the form (2.2) and thus also an associated family of constant mean curvature tori.

Theorem 3 ([14], Theorem 8.1). *Let $\tilde{\Sigma}$ be a (possibly singular) hyperelliptic curve over \mathbb{CP}^1 defined by the equation*

$$\eta^2 = \lambda \prod_{i=1}^p \bar{q}_i^{-1} (\lambda - q_i) (\lambda - \bar{q}_i^{-1}),$$

And let

$$\Psi : T^2 \rightarrow \text{Pic}_{p+1}(\Sigma), x \mapsto \mathcal{L}_x$$

be a group homomorphism. Suppose all $\mathcal{L}_x \in \Psi(T^2)$ are non special and $\rho^* \mathcal{L}_x = \mathcal{L}_x \cdot \mathbb{J}$. Further, suppose $\mathcal{L}_x^* \otimes \sigma^* \mathcal{L}_x^* = L(2p+2)$, where $L(2p+2)$ is the bundle of degree $2p+2$ on \mathbb{CP}^1 .

Then we can construct a family of flat connections ∇^λ on W for which the lines $\mathcal{L}|_{(\eta, \lambda)}$ and $\mathcal{L}|_{(-\eta, \lambda)}$ define a parallel frame of W w.r.t. ∇^λ . This family has the form

$$\nabla^\lambda = \nabla + \frac{1}{2}(1 + \lambda^{-1})\alpha' + \frac{1}{2}(1 + \lambda)\alpha'',$$

for some unitary connection ∇ .

3. THE CONSTRAINED WILLMORE CASE

In order to define the associated family of flat connections for constrained Willmore tori, we recall some facts on quaternionic surface theory developed in [10]. The conformal geometry of the 4-sphere is modeled by the projective geometry of the quaternionic projective line \mathbb{HP}^1 . A map $f : T^2 \rightarrow S^4 \cong \mathbb{HP}^1$ is given by a quaternionic line subbundle $L := f^* \mathcal{T}$ of the trivial \mathbb{H}^2 -bundle $V := T^2 \times \mathbb{H}^2$, where \mathcal{T} is the tautological bundle of \mathbb{HP}^1 . Using affine coordinates we can consider

$$S^3 \subset \mathbb{H} \hookrightarrow \mathbb{HP}^1, \lambda \mapsto [\lambda, 1].$$

An oriented round 2-sphere in S^4 is given by a linear map $S : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ with $S^2 = -1$, i.e., a complex structure on \mathbb{H}^2 . For given f there exist a natural map S from the torus into the space of oriented 2-spheres - the conformal Gauß map. It assigns to every point p of the surface a 2-sphere S_p through the point $f(p)$ with the same tangent plane and the same mean curvature. The pair (V, S) is then a complex quaternionic vector bundle. The trivial connection $\hat{\nabla}$ on V splits into a complex linear part $\hat{\nabla}'$ and a complex anti-linear part $\hat{\nabla}''$ with respect to S . We denote by A and Q the S -anti-commuting parts of $\hat{\nabla}'$ and $\hat{\nabla}''$ respectively.

Theorem 4 ([5]). *Let $f : T^2 \rightarrow S^3$ be a conformal immersion. It is constrained Willmore if there exist a 1-form $\nu \in \Omega^1(\mathcal{R})$ such that*

$$d^{\hat{\nabla}}(2 * A + \nu) = 0,$$

where $\mathcal{R} = \{B \in \text{End}(V) \mid \text{Im} B \subset L \subset \text{Ker} B\}$.

The 1-form ν is the Lagrange multiplier. It is uniquely determined unless the immersion is isothermic. Let A_0 be the 1-form defined by $2 * A_0 = 2 * A + \nu$. Consider V as a complex rank 4 bundle with complex structure i given by the left multiplication of the quaternionic i together with an anti-linear endomorphism j which is the right multiplication by the quaternion j . The Euler-Lagrange equation for constrained Willmore tori is equivalent to the flatness of the associated family of $SL(4, \mathbb{C})$ -connections

$$\nabla^\mu = \hat{\nabla} + (\mu - 1)\frac{1 - iS}{2}A_0 + (\mu^{-1} - 1)\frac{1 + iS}{2}A_0,$$

for $\mu \in \mathbb{C}_*$. This associated family ∇^μ has the symmetry

$$\nabla^{\bar{\mu}^{-1}} = j^{-1}\nabla^\mu j.$$

Thus for all $\mu \in S^1 \subset \mathbb{C}_*$ the connection ∇^μ is quaternionic.

For a fixed point $x \in T^2 = \mathbb{C}/\Gamma$ consider the holonomy representations H_x^μ of the associated family ∇^μ . The representations are holomorphic in μ and the holonomy H_x^μ to different basis points are conjugated. Thus the eigenvalues are independent of $x \in T^2$. Since the first fundamental group of T^2 is abelian, we get that every simple eigenspace of $H_x^\mu(\gamma_0)$ for $\gamma_0 \in \Gamma$ is a simultaneous eigenspace for all $\gamma \in \Gamma$.

Theorem 5 ([5], Proposition 3.1 and Theorem 5.1). *Let $f : T^2 \rightarrow S^3$ be a constrained Willmore torus. Then the holonomy representation of the associated family ∇^μ belongs to the following 2 cases:*

- (1) *For $\gamma \in \Gamma$ the holonomy $H_x^\mu(\gamma)$ has 4 distinct eigenvalues for generic $\mu \in \mathbb{C}_*$. These eigenvalues are non-constant in μ .*
- (2) *All holonomies $H_x^\mu(\gamma)$ have a 2-dimensional common eigenspace with eigenvalue 1 and $H_p^\mu(\gamma)$ has 2 distinct and non constant eigenvalues for generic $\mu \in \mathbb{C}_*$.*

The spectral curve Σ of a constrained Willmore torus is the normalization and compactification of the 1-dimensional analytic variety

$$\{(\eta, \mu) \in \mathbb{C}_* \times \mathbb{C}_* | f(\eta, \mu) = 0\} \quad \text{with} \quad f(\eta, \mu) = \det(H_p^\mu(\gamma) - \eta Id).$$

This spectral curve differs from the one defined in [6], $\hat{\Sigma}$, only by two points which we denote by 0 and ∞ . These points corresponds to the ends of $\hat{\Sigma}$. It is a 4-fold covering of \mathbb{CP}^1 , if the holonomy representation of ∇^μ belongs to case (1) and it is hyperelliptic if the holonomy representation of ∇^μ belongs to case (2). We restrict ourselves in the following to the case, where the spectral curve is a 4-fold covering of \mathbb{CP}^1 since the second case only occurs for CMC surfaces by [5], Proposition (6.12).

We have two involutions ρ and σ on the spectral curve. Because of the symmetry j of ∇^μ the involution ρ covers the involution $\mu \mapsto \bar{\mu}^{-1}$ on \mathbb{CP}^1 . By the following lemma the involution σ fixes the spectral parameter μ .

Lemma 1. *Let $f : T^2 \rightarrow S^3$ be a constrained Willmore torus and ∇^μ the corresponding associated family of flat connections. Further let $H_x^\mu(\gamma)$ denote the holonomy of ∇^μ at a base point $x \in T^2$ along $\gamma \in \Gamma$. If η is an eigenvalue of $H_x^\mu(\gamma)$ then η^{-1} is also an eigenvalue of $H_x^\mu(\gamma)$.*

Proof. In the quaternionic setup the conformal 3-sphere in $\mathbb{H}P^1$ is given by the null lines of the indefinite product $(.,.)$ on \mathbb{H}^2

$$(v, w) = \bar{v}_1 w_2 + \bar{v}_2 w_1.$$

The \mathfrak{i} -anti-commuting part of $(.,.)$ gives rise to a symplectic form ω on \mathbb{H}^2 . Consider the bundle $L^\perp \subset V^*$, where V^* is a complex quaternionic bundle with respect to the complex structure $-i$. We can define a family of connections $(\nabla^\perp)^\mu$, for $\mu \in \mathbb{C}_*$ on L^\perp by

$$(\nabla^\perp)^\mu = \nabla^\perp + (\mu - 1) \frac{1 + iS^\perp}{2} A_0^\perp + (\mu^{-1} - 1) \frac{1 - iS^\perp}{2} A_0^\perp.$$

This family is dual to the family of connections

$$\tilde{\nabla}^\mu = \nabla + (\mu - 1) Q_0 \frac{1 - iS}{2} + (\mu^{-1} - 1) Q_0 \frac{1 + iS}{2},$$

with respect to $(.,.)$, since $A^\perp = -Q^*$, $Q^\perp = -A^*$ and $\nu^\perp = -\nu^*$.

Both families of connections are flat if and only if the immersion is constrained Willmore. By duality of these families we have: If η is an eigenvalue of the holonomy of $(\nabla^\perp)^\mu$ then η^{-1} is an eigenvalue of the holonomy of $\tilde{\nabla}^\mu$. On other hand, we have for conformal immersions into S^3 that $L = L^\perp$ and $S = S^* = S^\perp$ and thus $A = -Q^*$. Therefore $(\nabla^\perp)^\mu = \nabla^\mu$. Further, ∇^μ is gauge equivalent to $\tilde{\nabla}^\mu$. The gauge is given by

$$g = \frac{1}{2}((\mu + 1) - i(\mu - 1)S).$$

This proves the lemma. \square

For a fixed $x \in T^2$ consider the symplectic form ω on V_x defined in Lemma 1. At a generic point $(\eta, \mu) \in \Sigma$ there exist a basis of \mathbb{C}^4 such that $H_x^\mu(\gamma)$ is given by

$$H_x^\mu(\gamma) = \begin{pmatrix} \eta & 0 & 0 & 0 \\ 0 & \eta^{-1} & 0 & 0 \\ 0 & 0 & \tilde{\eta} & 0 \\ 0 & 0 & 0 & \tilde{\eta}^{-1} \end{pmatrix}.$$

Let $\tilde{\mathcal{L}}_x \subset \Sigma \times \mathbb{C}^4$ be the line bundle which at generic points $(\eta, \mu) \in \Sigma$ coincides with the eigenspace of $H_x^\mu(\gamma)$ to the eigenvalue η . Then the line bundle $g(\sigma^* \tilde{\mathcal{L}}_x)$ corresponds generically to the eigenspace of the holonomy of $\tilde{\nabla}^\mu$ w.r.t. the eigenvalue η^{-1} . The evaluation of ω on $\tilde{\mathcal{L}}_x \otimes g(\sigma^* \tilde{\mathcal{L}}_x)|_{(\eta, \mu)}$ is non-degenerate at generic points, where we have 4 distinct eigenvalues, and thus it defines a holomorphic map

$$\tilde{\mathcal{L}}_x \otimes g(\sigma^* \tilde{\mathcal{L}}_x) \rightarrow \mathbb{C}.$$

This map vanishes, if and only if the lines $\tilde{\mathcal{L}}_x$ and $g(\sigma^* \tilde{\mathcal{L}}_x)$ coalesce at (η, μ) . This is independent on x . The bundles $g(\sigma^* \tilde{\mathcal{L}}_x)$ and $\sigma^* \tilde{\mathcal{L}}_x$ are holomorphic isomorphic. Therefore we obtain

Lemma 2. *The bundle $\tilde{\mathcal{L}}_x \otimes \sigma^* \tilde{\mathcal{L}}_x$ is independent of $x \in T^2$ as a complex holomorphic line bundle.*

3.1. The reconstruction of a constrained Willmore immersion. Let Σ be a connected compact Riemann surface with an anti-holomorphic involution ρ . Further fix a real subtorus $Z = \Psi(T^2)$ of dimension 0, 1 or 2 of the Jacobian of Σ . For $x \in T^2$ the line bundle $\tilde{\mathcal{L}}_x$ over Σ is by construction a complex holomorphic line subbundle of $\Sigma \times \mathbb{C}^4$. Thus it defines a map from Σ to \mathbb{CP}^3 . A quaternionic structure on \mathbb{C}^4 is a real linear endomorphism \mathbf{j} with $\mathbf{j}^2 = -1$ anti-commuting with i . By fixing such a quaternionic structure \mathbf{j} on $\Sigma \times \mathbb{C}^4$ we obtain a canonical isomorphism between \mathbb{C}^4 and \mathbb{H}^2 . This isomorphism induces a map $\pi_{\mathbb{H}}$ between \mathbb{CP}^3 and \mathbb{HP}^1 which is called twistor projection. In our case the quaternionic structure is given by ρ . The main theorem for the reconstruction of a conformally immersed torus is the following.

Theorem 6 ([6], Theorem 4.2). *Let $f : T^2 \rightarrow S^3$ be a conformal immersion whose spectral curve Σ has finite genus. Then there exist a map*

$$(3.1) \quad F : T^2 \times \Sigma \rightarrow \mathbb{CP}^3,$$

such that

- $F(x, -) : \Sigma \rightarrow \mathbb{CP}^3$ is an algebraic curve, for all $x \in T^2$.
- The original conformal immersion $f : T^2 \rightarrow S^3$ is obtained by the twistor projection of the evaluation of F at the points at infinity:

$$f = \pi_{\mathbb{H}} F(-, 0) = \pi_{\mathbb{H}} F(-, \infty).$$

For given Σ with marked points 0 and ∞ and a T^2 -family of holomorphic line bundles Z in $\text{Jac}(\Sigma)$ the map F is in general not unique. In other words, the immersion f is in general not uniquely determined by the spectral curve and Z .

Let E be a complex holomorphic line bundle over a Riemann surface M . By the Kodaira embedding theorem there exist a holomorphic map s from E to \mathbb{CP}^n if and only if the space of holomorphic sections of the line bundle E^* is at least complex $(n + 1)$ -dimensional, i.e., $\deg(E^*) \geq n + 1$. The space of holomorphic sections of E^* is exactly $(n + 1)$ -dimensional if and only if the map s is unique up to a $PSL(n + 1, \mathbb{C})$ action on \mathbb{CP}^n . In our case we have $n = 4$ and the line bundles in Z must be compatible with the quaternionic structure \mathbf{j} . Elements of $PSL(4, \mathbb{C})$ compatible with \mathbf{j} act on \mathbb{HP}^1 as Möbius transformations. Thus $f = \pi_{\mathbb{H}}(F(-, 0))$ is uniquely determined up to Möbius transformations of S^4 if and only if the space of holomorphic sections of $\tilde{\mathcal{L}}_x$ is 4-dimensional.

Definition. A conformal immersion is simple if it is uniquely determined by Σ and Z up to Möbius transformations of S^4 .

Instead of the quaternionic line bundle L , we can also consider the quaternionic line bundle V/L . Let S denote the conformal Gauß map. The property $f(p) \in S_p$ translated into the quaternionic language is $S_p L_p \subset L_p$. Thus the projection of S to V/L defines a complex structure and the projection of $\hat{\nabla}''$ defines a holomorphic structure on V/L . By definition the projection of a constant sections of V to V/L is holomorphic. The map $f : T^2 \rightarrow S^3$ is then given by the quotient of these sections, since

$$\pi_{V/L}(0, 1) + \pi_{V/L}(1, 0)f = 0_{V/L}.$$

The next Lemma states that the only holomorphic sections of V/L comes from the constant sections of V , if f is simple. In this case the immersion is uniquely determined by its quotient bundle up to Möbius transformations of S^4 .

Lemma 3. *Let $f : T^2 \rightarrow S^3$ be a simple conformal immersion and V/L be the associated quotient bundle, then the space $H^0(V/L)$ is quaternionic 2-dimensional.*

Proof. If $H^0(V/L) > 2$ then there exist at least 3 quaternionic linear independent holomorphic sections. The quotient of any two of them yield a map from T^2 to $\mathbb{H}P^1$. Thus in this case we get at least two maps f and \tilde{f} which are not Möbius equivalent. The corresponding quotient bundles V/L and V/\tilde{L} are holomorphic isomorphic. Thus f and \tilde{f} have the same spectral curve and Z and the map F cannot be unique. \square

Theorem 7. *Let $f : T^2 \rightarrow S^3$ be a simple constrained Willmore immersion with spectral curve Σ . On Σ we have a fix point free anti-holomorphic involution $\rho : (\eta, \mu) \mapsto (\bar{\eta}, \bar{\mu}^{-1})$ and an involution $\sigma : (\eta, \mu) \mapsto (\eta^{-1}, \mu)$. If the quotient $\Sigma/\sigma \cong \mathbb{C}P^1$ and $\rho \circ \sigma$ has fixpoints, then f is a CMC torus in a space form.*

Proof. Let $f : T^2 \rightarrow S^3$ be a simple, conformal and constrained Willmore immersion. Let ∇^μ be its associated family of complex flat connections of constrained Willmore surfaces on the bundle $V = T^2 \times \mathbb{H}^2 \cong T^2 \times \mathbb{C}^4$. The spectral curve Σ is a 4-fold covering of $\mathbb{C}P^1$ and has two marked points 0 and ∞ corresponding to the ends of the spectral curve $\hat{\Sigma}$. These points are by [5] fixed under the involution σ and interchanged by the involution ρ . By assumption the quotient Σ/σ is also $\mathbb{C}P^1$. Thus the spectral curve Σ is a hyperelliptic curve and σ is the hyperelliptic involution. Let λ be a holomorphic coordinate of Σ/σ such that $0 \in \Sigma$ is the point over $\lambda = 0$ and $\infty \in \Sigma$ is the point over $\lambda = \infty$. Since the involutions ρ and σ commute and ρ interchanges the points 0 and ∞ on Σ , ρ also interchanges the points $\lambda = 0$ and $\lambda = \infty$ on Σ/σ . An anti-holomorphic involution on $\mathbb{C}P^1$ interchanging $\lambda = 0$ and $\lambda = \infty$ is either the map $\lambda \mapsto \bar{\lambda}^{-1}$ or the map $\lambda \mapsto -\bar{\lambda}^{-1}$. Since $\rho \circ \sigma$ has fixed points, ρ induces the involution $\lambda \mapsto \bar{\lambda}^{-1}$ on Σ/σ , which fixes the points over $\lambda \in S^1$.

By assumption there exist a T^2 -family of complex holomorphic line subbundles $\tilde{\mathcal{L}}_x$ of $\tilde{V} = \Sigma \times \mathbb{C}^4$. Therefore we obtain by Theorem 6 there is a map

$$F : T^2 \times \Sigma \rightarrow \mathbb{C}P^3,$$

such that the line bundle L corresponding to the immersion f can be reconstructed by $L = \pi_{\mathbb{H}} F(-, \infty)$. Thus we can define the quotient bundle V/L and the projection of $\tilde{\mathcal{L}}_x$, $x \in T^2$, to V/L defines a T^2 -family of complex line subbundles \mathcal{L}_x of the

topologically trivial complex rank 2 bundle V/L . We want to show that for fixed $x \in T^2$ the bundle \mathcal{L}_x^* has degree $g + 1$. By [6] the map

$$\Psi : T^2 \rightarrow \text{Jac}(\Sigma), x \mapsto \mathcal{L}_x \mathcal{L}_{x_0}^{-1}$$

is a group homomorphism for fixed $x_0 \in T^2$. Thus the degree of \mathcal{L}_x is constant in x . Moreover, since f is simple, the map F is unique up to Möbius transformations of S^4 . Thus the complex holomorphic bundle $\pi_L(F(x, -)) = \mathcal{L}_x \subset \Sigma \times (V/L)_x$ is unique up to Möbius transformations of \mathbb{CP}^1 and therefore the space of holomorphic sections of \mathcal{L}_x^* is 2-dimensional by the Kodaira embedding theorem.

Let $(\widetilde{\cdot}, \cdot)$ denote the parallel Hermitian form on V/L with respect to $\pi_{V/L}(\hat{\nabla})$. Then the i -anti-commuting part of $(\widetilde{\cdot}, \cdot)$ defines a symplectic structure $\hat{\omega}$ on V/L . The evaluation of $\hat{\omega}$ on $\mathcal{L}_x \otimes \sigma^* \mathcal{L}_x$ is non-degenerate at generic points on Σ and vanishes at the branch points of σ . Thus the degree d of \mathcal{L}_x^* satisfies $(g + 1) \leq d$, where g is the genus of Σ . By the Riemann-Roch theorem we thus obtain that the line bundle \mathcal{L}_x^* is non-special and $\deg(\mathcal{L}_x^*) = g + 1$. Using Theorem 3, we can therefore define a family of flat connections ∇^λ , $\lambda \in \mathbb{C}_*$ on V/L of the form

$$\nabla^\lambda = \nabla + \frac{1}{2}(1 + \lambda^{-1})\alpha' + \frac{1}{2}(1 + \lambda)\alpha'',$$

for which $(\pi_{V/L}(F(-, h)), \pi_{V/L}(F(-, \sigma(h))))$ define a parallel frame of the complex rank two bundle V/L .

The section $\psi_h = \pi_{V/L}(F(-, h))$ is by definition a holomorphic section with monodromy of V/L . Since f is simple, all holomorphic sections of V/L with trivial monodromy are given by the projection of constant sections of V by Proposition 3. Constant sections of V are parallel sections of $\hat{\nabla} = \nabla^{\mu=1}$, where ∇^μ is the constrained Willmore associated family of flat connections. This is a complex 4 dimensional space. Thus there exist at most 2 trivial connections in the ∇^λ family of flat connections. Let λ_0 and λ_1 denote the points of Σ/σ over $\mu = 1$. There are three cases to consider.

In the first case both points $\lambda_0 \neq \lambda_1$, are fixed under the involution $\rho \circ \sigma$, then λ_0 and λ_1 has length 1. Thus the reconstruction by the Sym-Bobenko formula stated in Theorem 2 gives a CMC immersion in S^3 . In the second case $\lambda_0 = \lambda_1$ and we obtain a CMC immersion in \mathbb{R}^3 . In the third case we have again $\lambda_0 \neq \lambda_1$, but both points are not fixed under $\rho \circ \sigma$. The reconstruction by the Sym-Bobenko formula we obtain a CMC immersion in H^3 .

It remains to show that the immersion reconstructed by the Sym-Bobenko formula \tilde{f} is Möbius equivalent to the original immersion f . This holds since by construction the quotient bundles V/\tilde{L} and V/L are holomorphic isomorphic. \square

Corollary 1. *A conformally immersed CMC torus in a 3-dimensional space form is simple if and only if its arithmetic spectral genus p equals its geometric spectral genus g .*

Proof. The degree of the line bundle \mathcal{L}_x is by definition $p + 1$. If $p > g$ we obtain by Riemann-Roch that the space of holomorphic sections of \mathcal{L}_x^* is at least 3-dimensional. Therefore the map F cannot be unique by the Kodaira embedding theorem. \square

Corollary 2. *Let f be a constrained Willmore conformal immersion in S^3 with even spectral genus and $\Sigma/\sigma \cong \mathbb{CP}^1$, then the involution $\rho \circ \sigma$ has fixed points. In particular, if f is simple then it is CMC in a space form.*

Proof. Let Σ be a hyperelliptic curve with two marked points 0 and ∞ and the two involutions σ and ρ . Suppose $\rho \circ \sigma$ has no fixed points. Then ρ induces a fixpoint free involution on \mathbb{CP}^1 interchanging the points 0 and ∞ . Let λ be the holomorphic coordinate on \mathbb{CP}^1 such that the point $0 \in \Sigma$ lies over $\lambda = 0$ and the point $\infty \in \Sigma$ lies over $\lambda = \infty$. Then ρ on $\Sigma/\sigma \cong \mathbb{CP}^1$ is given by

$$\lambda \mapsto -\bar{\lambda}^{-1}$$

In this coordinate Σ is given by

$$\eta^2 = \prod_{i=1}^{g+1} \bar{q}_i^{-1} (\lambda - q_i)(\lambda + \bar{q}_i^{-1}) =: P(\lambda),$$

where $q_i, -\bar{q}_i^{-1} \in \mathbb{C}$ and $0, \infty$ are the branch points of Σ . It is easy to compute that

$$P(-\bar{\lambda}^{-1}) = (-1)^{g+1} \bar{\lambda}^{-(2g+2)} \overline{P(\lambda)}.$$

Therefore the map ρ on Σ inducing the involution $\lambda \mapsto -\bar{\lambda}^{-1}$ is given by

$$(\eta, \lambda) \mapsto (\pm i^{g+1} \bar{\eta} \bar{\lambda}^{-(g+1)}, -\bar{\lambda}^{-1}).$$

Since g is even this cannot define an involution on Σ . \square

Proposition 3. *A simple immersion $f : T^2 \rightarrow S^3$ (not necessarily constrained Willmore) of spectral genus 1 is equivariant, i.e., it has a 1-parameter group of Möbius symmetries. If $\rho \circ \sigma$ has fixed points then it is CMC torus in a space form.*

Proof. By assumption the map F is unique. Thus the bundle $L = \pi_{\mathbb{H}} F(-, \infty)$ is unique up to Möbius transformations of S^4 . If we only consider immersions into a fixed $S^3 \subset S^4$, we obtain that this reconstruction is unique up to Möbius transformations of S^3 . Since the Jacobian of a torus is the torus itself and the set of line bundles compatible with ρ is only a S^1 , the map Ψ has a 1-dimensional kernel. Now let x be the direction in $T^2 = \mathbb{C}/\Gamma$ parametrizing the kernel of Ψ . Then the conformal maps $f(x, y)$ and $f(x + x_0, y)$ have the same spectral curve Σ and Z . Thus there is a Möbius transformation M_{x_0} of S^3 with

$$f(x + x_0, y) = M_{x_0} f(x, y).$$

The map $M : \mathbb{R} \rightarrow \text{Möb}(S^3)$ is a group homomorphism, thus f is equivariant.

The involution σ is branched over the two ends of the spectral curve by [5]. Thus by the Riemann-Hurwitz formula we obtain that there must be two other branch points and $\Sigma/\sigma \cong \mathbb{CP}^1$. \square

Remark 2. If $\rho \circ \sigma$ has no fixed points, then f is given by the rotation of an elastic figure eight curve in the hyperbolic plane, modeled by the upper half plane, around the x -axis.

Corollary 3. *A simple constrained Willmore immersion in S^3 of spectral genus 2 is either equivariant or CMC in a space form.*

Proof. By Riemann-Hurwitz formula the involution σ has either 2 or 6 branch points. If σ has 6 branch points then Σ/σ is \mathbb{CP}^1 . Because the genus is even, the involution $\rho \circ \sigma$ has fixed points. Therefore we get a CMC torus in a space form.

If σ has 2 branch points, then Σ/σ is a torus. Let $x_0, x \in T^2$. We have

$$\mathcal{L}_x = E_x \otimes \mathcal{L}_{x_0}.$$

Since the bundle $\mathcal{L}_x^* \otimes \sigma^* \mathcal{L}_x^*$ is independent of x by Lemma 2, we obtain

$$E_x \otimes \sigma^* E_x = \underline{\mathbb{C}}.$$

Thus \mathcal{L}_x lies in a affine translate of the Prym variety of Σ with respect to σ for all $x \in T^2$. Since the Prym variety is complex 1 dimensional and $\rho^* \mathcal{L}_x = \mathcal{L}_{x\mathbb{j}}$, the image of the map Ψ is real 1-dimensional and has a 1-dimensional kernel. By the same arguments as in Proposition 3 we obtain that f is equivariant. \square

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